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# Abelian sandpile model on the Husimi lattice of square plaquettes

Vl V Papoyan† and R R Shcherbakov‡§
Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia

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**Abstract.** An Abelian sandpile model is considered on the Husimi lattice of square plaquettes. Exact expressions for the distribution of height probabilities in the self-organized critical state are derived. The two-point correlation function for the sites deep inside the Husimi lattice is calculated exactly.

## 1. Introduction

In recent years, there has been considerable interest in different dynamical models which can evolve without any tuning of parameters to the *self-organized critical* (SOC) state.

The concept of self-organized criticality has been introduced by Bak *et al* [1], through simple cellular automaton models known as *sandpiles*, to explain the temporal and spatial scaling in dynamical dissipative systems.

Later on, Dhar and Majumdar, in a number of papers [2–4], studied the so-called *Abelian sandpile models* (ASMs) since they show a non-trivial analytically tractable example of the SOC behaviour. For these models some ensemble-average quantities have been calculated on the square lattice [2, 4–7].

In this paper, we consider ASM on the Husimi lattice of square plaquettes. One of the remarkable features of the Husimi- or Bethe-like lattices is the exact solvability of different spin, gauge and dynamical models defined on them [8–14]. One might also hope, by examining the form of exact solutions for these lattices which allow only a limited number of closed configurations, to predict a general behaviour for the lattice models of principal interest.

It is well known that the exact solutions obtained for different spin models on the Bethe lattice are equivalent to the Bethe–Peierls approximation and they give a more precise description of the critical behaviour of these models than the mean-field approximation technique [15]. In turn, the Husimi lattice can be considered as a next step in this list of approximations and, as has been shown by Monroe [10] for the two- and three-site interacting spin systems, this approximation improves results obtained on the Bethe lattice. In section 6 we give a table where we compare the distribution of height probabilities in the SOC state of ASM calculated on different lattices and again we see that the results obtained on the Husimi lattice are in good agreement with the known exact results on the square

† E-mail address: papoyanv@theor.jinrc.dubna.su
‡ E-mail address: shcher@thsun1.jinr.dubna.su
§ Permanent address: Department of Physics, Yerevan State University, Alek Manoukian St 1, Yerevan, 375049, Armenia.

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lattice [4, 6]. This can be explained by the presence of elementary loops in the Husimi lattice that reproduce the local structure of the usual lattices more precisely than the Bethe lattice. Hence, for most of the flat and three-dimensional lattices we can find suitable Husimi-like lattices and examine different models on them.

The outline of the paper is as follows. In the next section, we define the lattice and ASM on it. In section 3, we find recursion relations for the numbers of allowed configurations in the SOC state. In section 4, we compute exactly the distribution of height probabilities. In section 5, we calculate the two-point correlation function for the sites deep inside the lattice. In section 6, we give some concluding remarks.

## 2. Lattice and model

A pure Husimi tree of square plaquettes [16] can be constructed recurrently. As a basic building block, we will take an elementary plaquette (figure 1(*a*)). This basic block will be called the *first-generation branch*. To construct the *second-generation branch*, we will attach a single basic building block at each free site of the first-generation branch except the base site (root) (figure 1(*b*)). Continuing this process we develop higher-generation branches. Then, at the final step, we will take four *n*th-generation branches and connect their base sites by the elementary plaquette. As a result, we will get the graph with the coordination number $q = 4$.

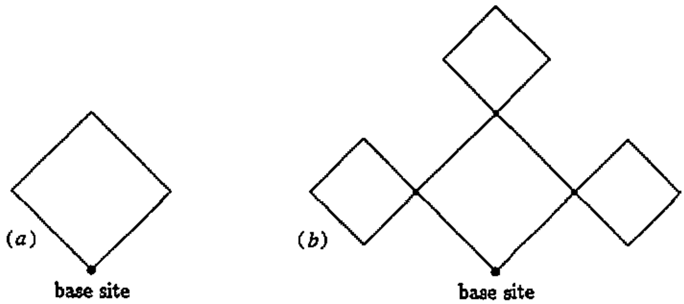


**Figure 1.** (*a*) A first-generation branch consists of a single square plaquette. (*b*) A second-generation branch.

Let us define ASM on this connected graph of $N$ sites as follows. To each site $i$ $(1 \leqslant i \leqslant N)$ we associate an integer $z_i$ $(1 \leqslant z_i \leqslant 4)$ which is the height of a column of sand grains. The evolution of the system is specified by two rules.

(i) Addition of a sand grain at a randomly chosen site $i$ increases $z_i$ by 1.

(ii) The site $i$ topples if the height $z_i$ exceeds the critical value $z_c = 4$ and sand grains drop on the nearest neighbours.

The number of surface sites of the Husimi tree is comparable with the interior ones. Hence, the calculation of the thermodynamic limit of the bulk properties requires special care. In our work, we define the height distribution of sand grains and the two-point correlation function for the sites deep inside the tree. Using these interior sites one can construct an infinite lattice, as they have the same features. Therefore, we will consider the problem on the Husimi lattice rather than on the Husimi tree.

Any configuration $\{z_i\}$ on the Husimi tree in which $1 \leqslant z_i \leqslant 4$ is a stable configuration under the toppling rule. These configurations can be divided into two classes: *allowed* and *forbidden* configurations [2].

In the SOC state, only allowed configurations have a non-zero probability. Any subconfiguration of heights $F$ on a finite connected set of sites is forbidden if

$$z_i \leqslant q_i \qquad \forall\, i \in F$$

where $q_i$ is a coordination number of a site $i$ in the given subconfiguration $F$ [2].

In turn, we can divide the allowed subconfigurations on an $n$th-generation branch of the Husimi tree into three non-overlapping classes: *weakly allowed of type 1* ($W_1$), *weakly allowed of type* 2 ($W_2$) and *strongly allowed* ($S$) subconfigurations.

Consider an allowed subconfiguration $C$ on the $n$th-generation branch $G_n$ with a root $a$ (figure 2($a$)). The coordination number of the root $a$ is $q = 2$. Adding a vertex $b$ to the $G_n$, one defines a subgraph $G' = G_n \cup b$. If the subconfiguration $C' = C \cup b$ with $z_b = 1$ on $G'$ is forbidden, then $C$ is called the weakly allowed subconfiguration of type 1 ($W_1$). Thus, $W_1$ can be locked by one bond, after which it becomes forbidden.

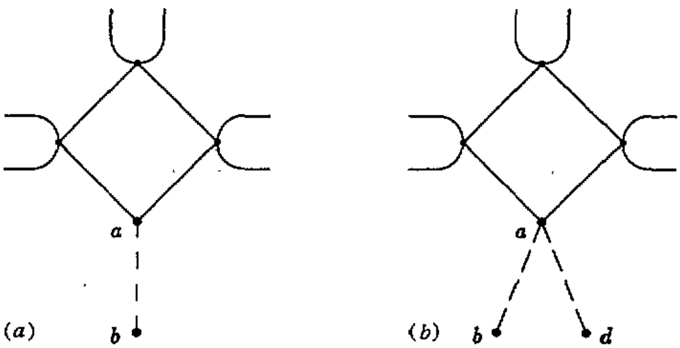


**Figure 2.** ($a$) An $n$th-generation branch $G_n$ and vertex $b$ form a subgraph $G'$. ($b$) Now two vertices $b$ and $d$ and the $G_n$ form a subgraph $G''$.

Now add two vertices $b$ and $d$ to $G_n$ and consider a subconfiguration $C'' = C \cup b \cup d$ on $G'' = G_n \cup b \cup d$ (figure 2($b$)). If the subconfiguration $C'' = C \cup b \cup d$ with $z_b = 1$ and $z_d = 1$ on $G''$ is forbidden, then $C$ is called the weakly allowed subconfiguration of type 2 ($W_2$).

Any allowed subconfigurations defined on the $n$th-generation branches that cannot be locked by one bond or by two bonds form a strongly allowed ($S$) class.

It is important to note that any subconfiguration of the $W_1$ type is also of the $W_2$ type. To obtain the non-overlapping classes, we always check first that the subconfiguration belongs to the $W_1$ type and only then to the $W_2$ type.

## 3. Recursion relations

In this section, we endeavour to find recursion relations for the numbers of allowed configurations on the $n$th-generation branch $G_n$ of the pure Husimi tree. Now consider $G_n$ with a root vertex $a$ (figure 3) that consists of three $(n-1)$th-generation branches $G^{(1)}_{n-1}$, $G^{(2)}_{n-1}$ and $G^{(3)}_{n-1}$ with roots $a_1$, $a_2$ and $a_3$, respectively. Let $N_{W_1}(G_n, i)$, $N_{W_2}(G_n, i)$ and $N_S(G_n, i)$ be the numbers of distinct $W_1$, $W_2$ and $S$-type subconfigurations on $G_n$ with a given height $z_a = i$ at the root $a$.

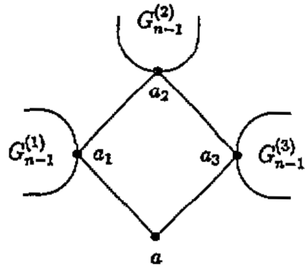


**Figure 3.** The $n$th-generation branch $G_n$ with three nearest $(n-1)$th-generation branches $G^{(1)}_{n-1}$, $G^{(2)}_{n-1}$ and $G^{(3)}_{n-1}$.

Let us also introduce

$$N_{W_1}(G_n) = \sum_{i=1}^{4} N_{W_1}(G_n, i) \tag{3.1}$$

$$N_{W_2}(G_n) = \sum_{i=1}^{4} N_{W_2}(G_n, i) \tag{3.2}$$

$$N_S(G_n) = \sum_{i=1}^{4} N_S(G_n, i)\,. \tag{3.3}$$

These numbers can be expressed in terms of the numbers of allowed subconfigurations on the three $(n-1)$th-generation branches $G^{(1)}_{n-1}$, $G^{(2)}_{n-1}$ and $G^{(3)}_{n-1}$:

$$\begin{aligned} N_{W_1}(G_n) = {} & N_SN_SN_S + N_{W_1}N_SN_S + N_{W_2}N_SN_S + N_SN_{W_1}N_S + N_{W_2}N_{W_1}N_S \\ & + N_SN_{W_2}N_S + N_{W_1}N_{W_2}N_S + N_{W_2}N_{W_2}N_S + N_SN_SN_{W_1} + N_{W_1}N_SN_{W_1} \\ & + N_{W_2}N_SN_{W_1} + N_SN_{W_2}N_{W_1} + N_{W_2}N_{W_2}N_{W_1} + N_SN_SN_{W_2} + N_{W_1}N_SN_{W_2} \\ & + N_{W_2}N_SN_{W_2} + N_SN_{W_1}N_{W_2} + N_{W_2}N_{W_1}N_{W_2} + N_SN_{W_2}N_{W_2} \\ & + N_{W_1}N_{W_2}N_{W_2} + N_{W_2}N_{W_2}N_{W_2} \end{aligned} \tag{3.4}$$

$$N_{W_2}(G_n) = N_{W_1}(G_n) \tag{3.5}$$

$$\begin{aligned} N_S(G_n) = {} & 2N_SN_SN_S + N_{W_1}N_SN_S + 2N_{W_2}N_SN_S + 2N_SN_{W_1}N_S + N_{W_2}N_{W_1}N_S \\ & + 2N_SN_{W_2}N_S + N_{W_1}N_{W_2}N_S + 2N_{W_2}N_{W_2}N_S + N_SN_SN_{W_1} + N_{W_2}N_SN_{W_1} \\ & + N_SN_{W_2}N_{W_1} + 2N_SN_SN_{W_2} + N_{W_1}N_SN_{W_2} + 2N_{W_2}N_SN_{W_2} \\ & + N_SN_{W_1}N_{W_2} + 2N_SN_{W_2}N_{W_2} \end{aligned} \tag{3.6}$$

where the first factor in each term of the sum corresponds to the $G^{(1)}_{n-1}$ branch, the second one to $G^{(2)}_{n-1}$ and the third one to $G^{(3)}_{n-1}$.

The fact that the numbers of the $W_1$-type subconfigurations and the $W_2$-type ones are equal to each other is seen from straightforward calculation.

Let us introduce

$$X = \frac{N_W}{N_S} \tag{3.7}$$

where $N_W \equiv N_{W_1} = N_{W_2}$.

If we consider graphs $G^{(1)}_{n-1}$, $G^{(2)}_{n-1}$ and $G^{(3)}_{n-1}$ to be isomorphic, then $N(G^{(1)}_{n-1}) = N(G^{(2)}_{n-1}) = N(G^{(3)}_{n-1})$ and from (3.4)–(3.7) one obtains the following recursion relation:

$$X(G_n) = \frac{1 + 4X(G_{n-1}) + 2X^2(G_{n-1})}{2[1 + 3X(G_{n-1})]}\,. \tag{3.8}$$

The iterative sequence $\{X(G_n)\}$ starts from the seed $X(G_0) = \frac{1}{2}$ and converges to the stable point $X^* = \frac{1+\sqrt{5}}{4}$ that characterizes, in the thermodynamic limit, the ratio of the $W_1$-type or the $W_2$-type configurations to the strongly allowed ones.

## 4. Distribution of height probabilities

One of the main characteristics that describes the SOC state is the probability $P(i)$ of having the height $z = i$ at a given site:

$$P(i) = \frac{N(i)}{N_{\text{total}}} \tag{4.1}$$

where $N(i)$ is the number of allowed configurations with a given value $z = i$ $(1 \leqslant i \leqslant 4)$ and $N_{\text{total}} = \sum_{i=1}^{4} N(i)$ is the total number of allowed configurations on the Husimi lattice.

Consider now a randomly chosen site $O$ deep inside the Husimi tree (figure 4). The number $N(i)$ can be expressed via the numbers of allowed configurations on the six $n$th-generation branches $G_n^{(\alpha)}$, $\alpha = 1, \ldots, 6$.

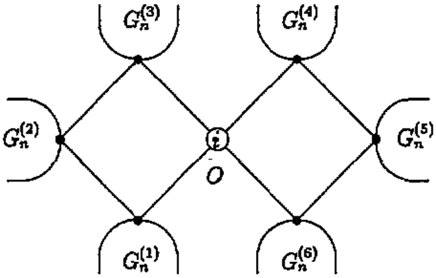


**Figure 4.** A site $O$ with height $i$ is located deep inside the lattice and surrounded by the six $n$th-generation branches $G_n^{(1)}, G_n^{(2)}, G_n^{(3)}, G_n^{(4)}, G_n^{(5)}$ and $G_n^{(6)}$.

If $i = 1$, then each allowed configuration on the branches $G_n^{(1)}$, $G_n^{(3)}$, $G_n^{(4)}$ and $G_n^{(6)}$ cannot be of $W_1$ type. It is also evident that three $W_2$-type configurations cannot occur on the neighbouring branches $G_n^{(1)}$, $G_n^{(2)}$, $G_n^{(3)}$, or $G_n^{(4)}$, $G_n^{(5)}$, $G_n^{(6)}$, and so on.

Excluding all such forbidden subconfigurations one finds the following expression for the number $N(1)$ of allowed configurations with $i = 1$ at $O$ for isomorphic branches $G_n^{(\alpha)}$, $\alpha = 1, \ldots, 6$:

$$N(1) = [1 + 8X + 22X^2 + 24X^3 + 9X^4] \prod_{\alpha=1}^{6} N_S(G^{(\alpha)}) . \tag{4.2}$$

Arguing similarly, one can get

$$N(2) = [1 + 12X + 50X^2 + 84X^3 + 45X^4] \prod_{\alpha=1}^{6} N_S(G^{(\alpha)}) \tag{4.3}$$

$$N(3) = [1 + 12X + 56X^2 + 124X^3 + 119X^4 + 24X^5] \prod_{\alpha=1}^{6} N_S(G^{(\alpha)}) \tag{4.4}$$

$$N(4) = [1 + 12X + 56X^2 + 128X^3 + 147X^4 + 72X^5] \prod_{\alpha=1}^{6} N_S(G^{(\alpha)}) . \tag{4.5}$$

For the sites far from the surface in the thermodynamic limit ($n \to \infty$) we have $X = \frac{1+\sqrt{5}}{4}$. Thus, from (4.1)–(4.5) we get

$$P(1) = \frac{5(\sqrt{5}-2)}{16} \qquad P(2) = \frac{10-3\sqrt{5}}{16}$$
$$P(3) = \frac{3(4-\sqrt{5})}{16} \qquad P(4) = \frac{4+\sqrt{5}}{16}. \tag{4.6}$$

## 5. Two-point correlation function.

The two-point correlation function may be defined as the probability $P_n(i, j)$ of a stable configuration in which two sites separated by the distance $n$ have heights $i$ and $j$. To obtain $P_n(i, j)$ explicitly, we consider two sites $A_1$ and $A_{n+1}$ deep inside the Husimi lattice (figure 5). The left-hand side of the lattice beginning from the vertex $A_k$, $k = 1, \ldots, n$, will be denoted as branch or subtree $G_k$ with the root $A_k$. This branch, in turn, consists of the subbranches $G_{k-1}$, $U_k^{(b)}$, $U_{k-1}^{(b)}$.

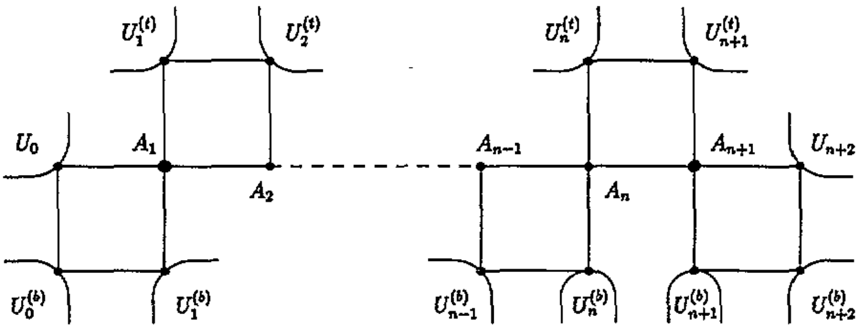


**Figure 5.** The path from the site $A_1$ to the site $A_{n+1}$ on the Husimi lattice goes through the points $A_2, \ldots, A_n$. The left-hand side of the lattice beginning from the vertex $A_k$, $k = 1, \ldots, n$, is denoted as a branch $G_k$ with the root $A_k$.

Following Dhar and Majumdar [3], we can solve the problem by the transfer matrix technique based on the fractal structure of the lattice.

The number of allowed configurations $N(i, j)$ on the Husimi lattice with fixed heights $i$ at $A_1$ and $j$ at $A_{n+1}$ is expressed via $\bar{N}_\alpha(G_n)$, $N_\alpha(U_n^{(t)})$, $N_\alpha(U_{n+1}^{(t)})$, $N_\alpha(U_{n+2})$, $N_\alpha(U_{n+2}^{(b)})$ and $N_\alpha(U_{n+1}^{(b)})$, where $\alpha = W_1, W_2, S$. The bars above $\bar{N}_\alpha(G_n)$ denote the constraint that the height at $A_1$ is fixed at $i$.

The numbers $\bar{N}_S(G_n)$, $\bar{N}_{W_1}(G_n)$ and $\bar{N}_{W_2}(G_n)$, in turn, can be expressed in terms of the numbers of allowed configurations on the branches $U_n^{(b)}$, $U_{n-1}^{(b)}$ and $G_{n-1}$ in the matrix form

$$\begin{pmatrix} \bar{N}_{W_1}(G_n) \\ \bar{N}_{W_2}(G_n) \\ \bar{N}_S(G_n) \end{pmatrix} = N_S(U_n^{(b)}) \cdot N_S(U_{n-1}^{(b)}) \cdot \mathbf{A} \cdot \begin{pmatrix} \bar{N}_{W_1}(G_{n-1}) \\ \bar{N}_{W_2}(G_{n-1}) \\ \bar{N}_S(G_{n-1}) \end{pmatrix} \tag{5.1}$$

where

$$\mathbf{A} = \begin{pmatrix} 1+3X+X^2 & 1+4X+3X^2 & 1+4X+3X^2 \\ 1+3X+X^2 & 1+4X+3X^2 & 1+4X+3X^2 \\ 1+2X & 2+6X & 2+7X+4X^2 \end{pmatrix}. \tag{5.2}$$

In the thermodynamic limit for the sites deep inside the lattice we have $X = \frac{1+\sqrt{5}}{4}$. Hence, one can get

$$\begin{pmatrix} \bar{N}_{W_1}(G_n) \\ \bar{N}_{W_2}(G_n) \\ \bar{N}_S(G_n) \end{pmatrix} = \prod_k N_S(U_k^{(b)}) \cdot N_S(U_k^{(t)}) \cdot \mathbf{A}^{n-1} \cdot \begin{pmatrix} N_{W_1}(G_1, i) \\ N_{W_2}(G_1, i) \\ N_S(G_1, i) \end{pmatrix} \tag{5.3}$$

where

$$\begin{pmatrix} N_{W_1}(G_1, i) \\ N_{W_2}(G_1, i) \\ N_S(G_1, i) \end{pmatrix} = \begin{pmatrix} \bar{N}_{W_1}(G_1) \\ \bar{N}_{W_2}(G_1) \\ \bar{N}_S(G_1) \end{pmatrix} \tag{5.4}$$

and the product runs over all subbranches of the subtree $G_n$.

To obtain the two-point correlation function, we ought to divide $N(i, j)$ by the total number of allowed configurations of the full lattice in the SOC state

$$P_n(i, j) = \frac{N(i, j)}{N_{\text{total}}} \tag{5.5}$$

where $N_{\text{total}}$ is given by

$$N_{\text{total}} = \frac{521 + 233\sqrt{5}}{2} N_S(G_n) N_S(U_n^{(t)}) N_S(U_{n+1}^{(t)}) N_S(U_{n+2}) N_S(U_{n+2}^{(b)}) N_S(U_{n+1}^{(b)}) \tag{5.6}$$

with

$$N_S(G_n) = \sum_{i=1}^{4} \bar{N}_S(G_n)\,. \tag{5.7}$$

Then, substituting $\bar{N}_S(G_n)$ from (5.3) one can find for $n > 1$

$$N_{\text{total}} = \frac{521\sqrt{5} + 1165}{40} \left(18\sqrt{5} + 40 + (3 + \sqrt{5})(\lambda_3 - \lambda_2)\right) {\lambda_3}^{n-1} \prod N_S(U) \tag{5.8}$$

where

$$\lambda_1 = 0 \tag{5.9}$$

$$\lambda_{2,3} = \frac{21 + 9\sqrt{5} \mp \sqrt{470 + 210\sqrt{5}}}{4} \tag{5.10}$$

are eigenvalues of the matrix $\mathbf{A}$ and the product runs over all subbranches which surround the path from $A_1$ to $A_{n+1}$.

Thus, after a rather tedious calculation we obtain an exact expression for $P_n(i, j)$ in the SOC state

$$P_n(i, j) = P(i)P(j) + p_{ij} \left(\frac{\lambda_3}{\lambda_2}\right)^{-(n-1)} \qquad n > 1 \tag{5.11}$$

where $p_{ij}$ are numerical constants

$$p_{11} = 8140 + 3640\sqrt{5} - \left(1282\sqrt{5} + 2870\right)(\lambda_3 - \lambda_2)$$

$$p_{12} = p_{21} = 50\,660 + 22\,656\sqrt{5} - \left(2762\sqrt{5} + 6174\right)(\lambda_3 - \lambda_2)$$

$$p_{13} = p_{31} = 122\,760 + 54\,900\sqrt{5} - \left(3234\sqrt{5} + 7230\right)(\lambda_3 - \lambda_2)$$

$$p_{14} = p_{41} = 171\,560 + 76\,724\sqrt{5} - \left(3026\sqrt{5} + 6766\right)(\lambda_3 - \lambda_2)$$

$$p_{22} = 175\,612 + 78\,536\sqrt{5} - \left(\frac{33\,498}{\sqrt{5}} + 14\,982\right)(\lambda_3 - \lambda_2)$$

$$p_{23} = p_{32} = 333\,040 + 148\,940\sqrt{5} - (9290\sqrt{5} + 20\,774)(\lambda_3 - \lambda_2)$$
$$p_{24} = p_{42} = 425\,488 + 190\,284\sqrt{5} - \left(\frac{49\,922}{\sqrt{5}} + 22\,326\right)(\lambda_3 - \lambda_2)$$
$$p_{33} = 521\,612 + 233\,272\sqrt{5} - \left(\frac{76\,682}{\sqrt{5}} + 34\,294\right)(\lambda_3 - \lambda_2)$$
$$p_{34} = p_{43} = 605\,724 + 270\,888\sqrt{5} - \left(\frac{91\,674}{\sqrt{5}} + 40\,998\right)(\lambda_3 - \lambda_2)$$
$$p_{44} = 662\,860 + 296\,440\sqrt{5} - \left(\frac{115\,466}{\sqrt{5}} + 51\,638\right)(\lambda_3 - \lambda_2).$$

For two nearest-neighbour sites $A_1$ and $A_2$ separated by the distance $n = 1$ in the SOC state the constants $p_{ij}$ have other values because the matrix **A** is degenerate. Hence we get

$$P_1(1,1) = 0 \qquad P_1(2,2) = \frac{175 - 78\sqrt{5}}{16}$$
$$P_1(3,3) = \frac{119 - 47\sqrt{5}}{128} \qquad P_1(4,4) = \frac{87 - 31\sqrt{5}}{128}$$
$$P_1(1,2) = P_1(2,1) = \frac{5(13\sqrt{5} - 29)}{32} \qquad P_1(2,3) = P_1(3,2) = \frac{169\sqrt{5} - 369}{128}$$
$$P_1(1,3) = P_1(3,1) = \frac{265 - 117\sqrt{5}}{128} \qquad P_1(2,4) = P_1(4,2) = \frac{171\sqrt{5} - 371}{128}$$
$$P_1(1,4) = P_1(4,1) = \frac{235 - 103\sqrt{5}}{128} \qquad P_1(3,4) = P_1(4,3) = \frac{81 - 29\sqrt{5}}{128}.$$

## 6. Conclusion

In this paper, we have investigated the Abelian sandpile model on the Husimi lattice of square plaquettes. The distribution of height probabilities and the two-point correlation function in the SOC state have been calculated exactly. We have shown that correlations decay with distance as $(\lambda_3/\lambda_2)^{-(n-1)}$.

At the end we want to compare the distribution of height probabilities obtained on different lattices with the same coordination number $q = 4$.

**Table 1.**

| | Square [6] | Bethe [3] | Husimi of triangles [14] | Husimi of squares |
|---|---|---|---|---|
| $P(1)$ | 0.073 63 | 0.074 07 | 0.070 31 | 0.073 77 |
| $P(2)$ | 0.173 9 | 0.222 22 | 0.164 06 | 0.205 74 |
| $P(3)$ | 0.306 3 | 0.333 33 | 0.335 94 | 0.330 74 |
| $P(4)$ | 0.446 1 | 0.370 37 | 0.429 69 | 0.389 75 |

From this table we see that the Husimi lattice is a good approximation for the regular lattices and the best one is achieved for $P(1)$ as it can be considered as a local structural characteristic of the model [6]. The next step of our investigations will be the calculation of dynamic characteristics of ASM, e.g. critical exponents of avalanches. The choice of the Husimi lattice gives us hope to find the relationship between the chaos and SOC state since some spin models formulated on this lattice show the chaotic behaviour [10, 13].

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